Übungsaufgaben XI, von 21.01.2002

Musterlösungen

1.

$$\tilde{\nu} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \quad \text{für die Paschen-Serie } n_1 = 3$$

$$\tilde{\nu} = \frac{1}{\lambda} = \frac{\nu}{c}$$

• für $n_2 = 4$

$$\Rightarrow \tilde{\nu} = 109677\,\text{cm}^{-1} \cdot \left(\frac{1}{9} - \frac{1}{16}\right) = 5331.51\,\text{cm}^{-1}$$

$$\Rightarrow \lambda = \frac{1}{5.3315 \cdot 10^5\,\text{m}^{-1}} = 1.8756 \cdot 10^{-6}\,\text{m} = 1876\,\text{nm}$$

$$\Rightarrow \nu = \frac{c}{\lambda} = \frac{2.998 \cdot 10^8\,\text{ms}^{-1}}{1.8756 \cdot 10^{-6}\,\text{m}} = 1.5984 \cdot 10^{14}\,\text{s}^{-1}$$

• für $n_2 = 5$

$$\Rightarrow \tilde{\nu} = 109677\,\text{cm}^{-1} \cdot \left(\frac{1}{9} - \frac{1}{25}\right) = 7799.25\,\text{cm}^{-1}$$

$$\Rightarrow \lambda = \frac{1}{7.7993 \cdot 10^5\,\text{m}^{-1}} = 1.2822 \cdot 10^{-6}\,\text{m} = 1282\,\text{nm}$$

$$\Rightarrow \nu = \frac{c}{\lambda} = \frac{2.998 \cdot 10^8\,\text{ms}^{-1}}{1.2822 \cdot 10^{-6}\,\text{m}} = 2.3382 \cdot 10^{14}\,\text{s}^{-1}$$

• für $n_2 = 6$

$$\Rightarrow \tilde{\nu} = 109677\,\text{cm}^{-1} \cdot \left(\frac{1}{9} - \frac{1}{36}\right) = \frac{1}{12} \cdot 109677\,\text{cm}^{-1} = 9139.75\,\text{cm}^{-1}$$

$$\Rightarrow \lambda = \frac{1}{9139.75\,\text{cm}^{-1}} = 1.0941 \cdot 10^{-6}\,\text{m} = 1094\,\text{nm}$$

$$\Rightarrow \nu = \frac{c}{\lambda} = \frac{2.998 \cdot 10^8\,\text{ms}^{-1}}{1.0941 \cdot 10^{-6}\,\text{m}} = 2.7402 \cdot 10^{14}\,\text{s}^{-1}$$

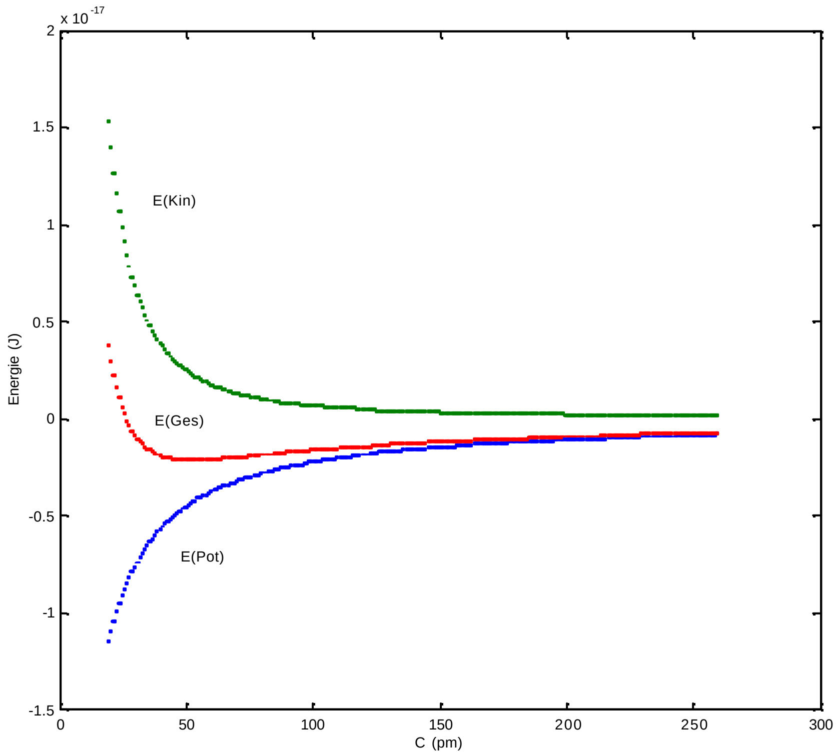
• für $n_2 = 7$

$$\Rightarrow \tilde{\nu} = 109677\,\text{cm}^{-1} \cdot \left(\frac{1}{9} - \frac{1}{49}\right) = 9948.03\,\text{cm}^{-1}$$

$$\Rightarrow \lambda = \frac{1}{9.9480 \cdot 10^5\,\text{m}^{-1}} = 1.0052 \cdot 10^{-6}\,\text{m} = 1005\,\text{nm}$$

$$\Rightarrow \nu = \frac{c}{\lambda} = \frac{2.998 \cdot 10^8\,\text{ms}^{-1}}{1.0052 \cdot 10^{-6}\,\text{m}} = 2.9825 \cdot 10^{14}\,\text{s}^{-1}$$

2.



$$E_{min} \quad \Rightarrow \quad \frac{dE}{dC} = 0$$

$$E = \frac{\hbar^2}{2mC^2} - \frac{e^2}{4\pi\varepsilon_0 C}$$

$$E = \frac{h^2}{8\pi^2 m_e C^2} - \frac{e^2}{4\pi\varepsilon_0 C}$$

$$\frac{dE}{dC} = -\frac{h^2}{4\pi^2 m_e C^3} + \frac{e^2}{4\pi\varepsilon_0 C^2} = 0$$

$$\Rightarrow \frac{e^2}{4\pi\varepsilon_0 C^2} = \frac{h^2}{4\pi^2 m_e C^3}$$

$$\Rightarrow \frac{e^2}{\varepsilon_0} = \frac{h^2}{\pi m_e} \cdot \frac{1}{C}$$

$$\Rightarrow C = \frac{h^2 \varepsilon_0}{\pi m_e e^2}$$

$$C = \frac{(6.626 \cdot 10^{-34} \cdot Js)^2 (8.854 \cdot 10^{-12} \cdot J^{-1} C^2 m^{-1})}{(3.14) \cdot (9.109 \cdot 10^{-31} \cdot kg) \cdot \left(1.6022 \cdot 10^{-19} \cdot C\right)^2} = 52.92 \cdot 10^{-12}\,m = 52.92pm$$

$$\hat{H}\psi(\tau) = E\psi(\tau)$$

$$\hat{H}\psi(\tau) = -\frac{\hbar^2}{2m}\nabla^2\psi(\tau) - \frac{e^2}{r}\psi(\tau)$$

$$\psi(\tau) = R(r)\cdot Y(\theta,\phi)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]R\cdot Y - \frac{e^2}{r}R\cdot Y = E\cdot R\cdot Y$$

$$\Rightarrow \frac{\hbar^2}{2m}\left[\frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\right]+\frac{e^2}{r}R + E\cdot R = -\frac{1}{Y}\frac{\hbar^2}{2mr^2}\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\phi^2}\right)\right]$$

$$\Rightarrow \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left(E+\frac{e^2}{r}\right) = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]$$

$$\frac{1}{R}\Delta_r R = \frac{1}{Y}\Delta_{\theta,\phi}Y$$

$$\Rightarrow \Delta_{\theta,\phi} = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

für $\quad Y_{0,0} = \sqrt{\frac{1}{4\pi}}$

$$\Delta_{\theta,\phi}Y_{0,0} = 0 \quad Y_{0,0} \neq f(\theta,\phi)$$

für $\quad Y_{1,1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$

$$\Delta_{\theta,\phi}Y_{1,1} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta e^{i\phi}\sqrt{\frac{3}{8\pi}}\right)+\frac{1}{\sin^2\theta}\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} =$$

$$= -\frac{1}{\sin\theta}\sqrt{\frac{3}{8\pi}}e^{i\phi}\left(\underbrace{\cos^2\theta}_{1-\sin^2\theta}-\sin^2\theta\right)+\frac{1}{\sin^2\theta}\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta =$$

$$= -\frac{1}{\sin\theta}\sqrt{\frac{3}{8\pi}}e^{i\phi}\left(1-2\sin^2\theta\right)+\frac{1}{\sin\theta}\sqrt{\frac{3}{8\pi}}e^{i\phi} =$$

$$= -\frac{1}{\sin\theta}\sqrt{\frac{3}{8\pi}}e^{i\phi} + 2\sin\theta\sqrt{\frac{3}{8\pi}}e^{i\phi} + \frac{1}{\sin\theta}\sqrt{\frac{3}{8\pi}}e^{i\phi} = 2\sin\theta\sqrt{\frac{3}{8\pi}}e^{i\phi} = 2Y$$

4.

$$\langle r_{nlm}\rangle = \iiint \psi^*_{nlm} r \psi_{nlm} r^2 \cdot dr \cdot \sin\theta \cdot d\theta \cdot d\phi = \int_0^\infty R^*_{nl}(r) r R_{nl}(r) r^2 \cdot dr \underbrace{\int_0^\pi \int_0^{2\pi} Y^*_{lm}(\theta,\phi) Y_{lm}(\theta,\phi) \cdot \sin\theta \cdot d\theta \cdot d\phi}_{=1}$$

$$\Rightarrow \langle r_{nl}\rangle = \int_0^\infty r^3 \left|R_{nl}(r)\right|^2 \cdot dr$$

für 1s

$$\langle r_{10}\rangle = \int_0^\infty r^3 \left(2\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0} \right)^2 \cdot dr = \frac{4Z^3}{a_0^3} \int_0^\infty r^3 e^{-2Zr/a_0} dr = \frac{4Z^3}{a_0^3} \cdot \left(\frac{3!}{16Z^4} a_0^4 \right) = \frac{3a_0}{2Z}$$

für 2s

$$\langle r_{20}\rangle = \int_0^\infty r^3 \left(\frac{1}{2\sqrt{2}} \left(\frac{Z}{a_0}\right)^{3/2} \left(2 - \frac{Zr}{a_0}\right) e^{-Zr/2a_0} \right)^2 \cdot dr = \frac{1}{8} \frac{Z^3}{a_0^3} \int_0^\infty r^3 \left(4 - \frac{4Zr}{a_0} + \frac{Z^2 r^2}{a_0^2} \right) e^{-Zr/a_0} dr =$$

$$= \frac{Z^3}{8a_0^3} \int_0^\infty \left(4r^3 - \frac{4Z}{a_0} r^4 + \frac{Z^2}{a_0^2} r^5 \right) e^{-Zr/a_0} dr = \frac{Z^3}{8a_0^3} \cdot \left(4 \frac{3!}{Z^4} a_0^4 - \frac{4Z}{a_0} \cdot \frac{4!}{Z^5} a_0^5 + \frac{Z^2}{a_0^2} \cdot \frac{5!}{Z^6} a_0^6 \right) =$$

$$= \frac{Z^3}{8a_0^3} \cdot \left(\frac{48a_0^4}{Z^4} \right) = \frac{6a_0}{Z}$$

für 2p

$$\langle r_{21}\rangle = \int_0^\infty r^3 \left(\frac{1}{2\sqrt{6}} \left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \right)^2 \cdot dr = \frac{1}{24} \frac{Z^3}{a_0^3} \int_0^\infty r^3 \frac{Z^2 r^2}{a_0^2} e^{-Zr/a_0} dr = \frac{Z^3}{24a_0^3} \frac{Z^2}{a_0^2} \int_0^\infty r^5 e^{-Zr/a_0} dr =$$

$$= \frac{Z^5}{24a_0^5} \cdot \left(\frac{5!}{Z^6} a_0^6 \right) = \frac{5a_0}{Z}$$