Übungsaufgaben IX, von 7.1.2002

Musterlösungen

1.

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$[\hat{A},\hat{B}]\psi = \left[\hat{A}\hat{B} - \hat{B}\hat{A}\right]\psi = \hat{A}\hat{B}\psi - \hat{B}\hat{A}\psi$$

$$\hat{x} = x.\,; \quad \hat{p}_x = -i\hbar\frac{\partial}{\partial x}$$

$$[\hat{x},\hat{p}_x]\psi = x.-i\hbar\frac{\partial}{\partial x}\psi + i\hbar\frac{\partial}{\partial x}(x\psi) = -i\hbar x\frac{\partial\psi}{\partial x} + i\hbar\psi + i\hbar x\frac{\partial\psi}{\partial x} = i\hbar\psi$$

$$\Rightarrow [\hat{x},\hat{p}_x] = i\hbar$$

$$\hat{p}_y = -i\hbar\frac{\partial}{\partial y}$$

$$[\hat{x},\hat{p}_y]\psi = x.-i\hbar\frac{\partial}{\partial y}\psi + i\hbar\frac{\partial}{\partial y}(x\psi) = 0$$

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2. Drehimpuls: $L = mrv = n\hbar$

$$F_c = F_z \qquad \Leftrightarrow \qquad \frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{r^2} = \frac{m_e v^2}{r}$$

wobei $\frac{m_e v^2}{r} = \frac{m_e rv \cdot m_e rv}{m_e r^3} = \frac{L^2}{m_e r^3} = \frac{n^2\hbar^2}{m_e r^3}$

$\Rightarrow r_n = \frac{4\pi\hbar^2\varepsilon_0}{Ze^2 m_e}\cdot n^2 = a_0 \cdot n^2$ wobei a_0 = Bohr'sche Atomradius

(fur Wasserstoffatom: Z = 1)

$$E_n = E_{kin} + E_{pot}$$

$E_{kin} = \frac{p^2}{2m_e} = \frac{n^2\hbar^2}{2m_e r^2}$ da $p = mv = \frac{L}{r} = \frac{n\hbar}{r}$

$$E_{pot} = -\int_r^{+\infty} F_C\,dr = -\int_r^{+\infty}\frac{Ze^2}{4\pi\varepsilon_0 r^2}dr = -\frac{Ze^2}{4\pi\varepsilon_0 r}$$

$$E_n = \frac{n^2\hbar^2}{2m_e}\cdot\left(\frac{Ze^2 m_e}{4\pi n^2\hbar^2\varepsilon_0}\right)^2 - \frac{Ze^2}{4\pi\varepsilon_0}\cdot\frac{Ze^2 m_e}{4\pi n^2\hbar^2\varepsilon_0} = \frac{Z^2e^4m_e}{32\pi^2n^2\hbar^2\varepsilon_0^2} - \frac{Z^2e^4m_e}{16\pi^2n^2\hbar^2\varepsilon_0^2}$$

$E_n = -\frac{Z^2e^4m_e}{32\pi^2\hbar^2\varepsilon_0^2}\cdot\frac{1}{n^2} = \frac{E_0}{n^2}$ wobei E_0 = -13.6 eV

3.

a) ψ_o^i Eigenfunktion von $\hat{O}$: $\hat{O}\,\psi_o^i = o_i \psi_o^i$, wobei o_i = Konstante

(i) $\frac{d}{dx} e^{ikx} = ike^{ikx} \Rightarrow \psi = e^{ikx}$ ist Eigenfunktion von $\frac{d}{dx}$ mit Eigenwert **ik**

$\frac{d}{dx}\cos kx = -k\sin kx \Rightarrow \psi = \cos kx$ ist keine Eigenfunktion von $\frac{d}{dx}$

$\frac{d}{dx} kx = k = \frac{1}{x} kx \Rightarrow\ ? = kx$ ist keine Eigenfunktion von $\frac{d}{dx}$, $\frac{1}{x} \neq$ Konstante

$\frac{d}{dx} e^{-ax^2} = -2axe^{-ax^2} \Rightarrow \psi = e^{-ax^2}$ ist keine Eigenfunktion von $\frac{d}{dx}$,
$-2ax \neq$ Konstante

(ii) $\frac{d^2}{dx^2} e^{ikx} = -k^2 e^{ikx} \Rightarrow \psi = e^{ikx}$ ist Eigenfunktion von $\frac{d^2}{dx^2}$ mit Eigenwert **-k²**

$\frac{d^2}{dx^2}\cos kx = -k^2 \cos kx \Rightarrow \psi = \cos kx$ ist Eigenfunktion von $\frac{d^2}{dx^2}$ mit Eigenwert **-k²**

$\frac{d^2}{dx^2} kx = 0 \Rightarrow \psi = kx$ ist Eigenfunktion von $\frac{d^2}{dx^2}$ mit Eigenwert **0**

$\frac{d^2}{dx^2} e^{-ax^2} = -2ae^{-ax^2} + (-2ax).(-2ax)e^{-ax^2} = (-2a + 4a^2x^2)e^{-ax^2} \Rightarrow \psi = e^{-ax^2}$ ist keine Eigenfunktion von $\frac{d^2}{dx^2}$, $-2a + 4a^2x^2 \neq$ Konstante

(iii) $\psi = e^{ikx}$ ist Eigenfunktion von $\frac{d}{dx}$ und $\frac{d^2}{dx^2}$

b) Vorlesung : $\left\langle \hat{O} \right\rangle = \frac{\int \psi^* \hat{O} \psi dx}{\int \psi^* \psi dx}$

$\hat{p} = -i\hbar \frac{d}{dx}$

Allgemeine Ausdruck für $\langle p_x \rangle$:

$$\langle p_x \rangle = \int \psi^* \hat{p} \psi dx = \frac{\int \psi^* \hat{p} \psi dx}{\int \psi^* \psi dx} = \frac{\int \psi^* \left(-i\hbar \frac{d\psi}{dx}\right) dx}{\int \psi^* \psi dx} = \frac{-i\hbar \int \psi^* \frac{d\psi}{dx} dx}{\int \psi^* \psi dx}$$

$$\text{für } \psi = e^{ikx} \qquad \frac{d\psi}{dx} = ik\psi$$

$$\langle p_x \rangle = \frac{-i\hbar \int \psi^* (ik)\psi dx}{\int \psi^* \psi dx} = -i\hbar \cdot ik \frac{\int \psi^* \psi dx}{\int \psi^* \psi dx} = -(i^2)\hbar k = \hbar k$$

$$\text{für } \psi = \cos kx \qquad \frac{d\psi}{dx} = -k \sin kx$$

$$\langle p_x \rangle = 0$$

$$\text{weil } \int_{-\infty}^{\infty} \psi^* \frac{d\psi}{dx} dx = -k \int_{-\infty}^{\infty} \cos(kx)\sin(kx) dx = 0$$

$$\text{für } \psi = e^{-ax^2} \qquad \frac{d\psi}{dx} = -2axe^{-ax^2}$$

$$\langle p_x \rangle = 0$$

$$\text{weil } \int_{-\infty}^{\infty} \psi^* \frac{d\psi}{dx} dx = -2a \int_{-\infty}^{\infty} \underbrace{xe^{-2ax^2}}_{\text{ungerade funktion}} dx = 0$$

$$\text{für } \psi = kx \qquad \frac{d\psi}{dx} = k$$

$$\langle p_x \rangle = 0$$

$$\text{weil } \int_{-\infty}^{\infty} \psi^* \frac{d\psi}{dx} dx = \int_{-\infty}^{\infty} kx \cdot k dx = k^2 \int_{-\infty}^{\infty} x dx = 0$$

4.

Für ein Teilchen in einem eindimensionalen Kasten der Länge L gilt:

$$\hat{E}\psi_E^n = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_E^n$$

$$\text{wobei } \quad \psi_E^n = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$$

$$\hat{p} = -i\hbar \frac{d}{dx}$$

$$\langle p \rangle = \int \psi^* \hat{p} \psi dx = \int \psi^* \left(-i\hbar \frac{d\psi}{dx} \right) dx$$

$$\langle p \rangle = -i\hbar \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \frac{d}{dx} \sin\left(\frac{n\pi x}{L}\right) dx = -i\hbar \frac{2}{L} \frac{n\pi}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0$$

$$\hat{p}^2 = -\hbar^2 \frac{d^2}{dx^2}$$

$$\langle p^2\rangle = -\hbar^2 \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\frac{d^2}{dx^2}\sin\left(\frac{n\pi x}{L}\right)dx = -\hbar^2\frac{2}{L}\frac{n\pi}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\frac{d}{dx}\cos\left(\frac{n\pi x}{L}\right)dx =$$

$$= \hbar^2\frac{2}{L}\left(\frac{n\pi}{L}\right)^2\int_0^L\sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{2\hbar^2}{L}\left(\frac{n\pi}{L}\right)^2\frac{1}{2}\int_0^L\left(1-\cos\frac{2n\pi x}{L}\right)dx = \frac{\hbar^2}{L}\left(\frac{n\pi}{L}\right)^2\left(x-\frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right)\Bigg|_0^L =$$

$$= \frac{\hbar^2}{L}\left(\frac{n\pi}{L}\right)^2 L = \frac{n^2h^2}{4L^2}$$

$$!\left(\sin^2 x = \frac{1-\cos 2x}{2}\right)!$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{n^2h^2}{4L^2}} = \frac{nh}{2L}$$

$$\langle x\rangle = \int \psi^* \hat{x}\psi dx$$

$$\langle x\rangle = \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)x..\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x\sin^2\left(\frac{n\pi x}{L}\right)dx =$$

$$= \frac{2}{L}\left(\frac{x^2}{4} - \frac{x\sin(2n\pi x/L)}{4n\pi/L} - \frac{\cos(2n\pi x/L)}{8(n\pi/L)^2}\right)\Bigg|_0^L = \frac{2}{L}\left(\frac{L^2}{4}\right) = \frac{L}{2}$$

$$\langle x^2\rangle = \frac{2}{L}\int_0^L x^2\sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x^2\,\frac{1-\cos(2n\pi x/L)}{2}dx = \frac{2}{L}\frac{1}{2}\int_0^L x^2\cdot(1-\cos(2n\pi x/L))dx =$$

$$= \frac{1}{L}\int_0^L x^2 dx - \int_0^L x^2\cos(2n\pi x/L)dx =$$

$$= \frac{1}{L}\left(\frac{x^3}{3} - \left(\frac{x^2}{2n\pi/L} - \frac{1}{4(n\pi/L)^3}\right)\sin(2n\pi x/L) - \frac{x\cos(2n\pi x/L)}{2(n\pi/L)^2}\right)\Bigg|_0^L =$$

$$= \frac{1}{L}\left(\frac{L^3}{3} - \frac{L^3}{2n^2\pi^2}\right) = L^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\left[L^2\left(\frac{1}{3}-\frac{1}{2n^2\pi^2}\right) - \frac{L^2}{4}\right]} = L\sqrt{\left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right)}$$

$$!\left(\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax\right)!$$

$$\Delta x \Delta p \geq \hbar / 2$$

$$\Delta x \Delta p = L\sqrt{\left(\frac{1}{12}-\frac{1}{2n^2\pi^2}\right)}.\frac{nh}{2L} = \frac{nh}{2}\sqrt{\left(\frac{1}{12}-\frac{1}{2n^2\pi^2}\right)} \overset{?}{>} \hbar / 2$$

$$\frac{nh}{2}\sqrt{\left(\frac{1}{12}-\frac{1}{2n^2\pi^2}\right)} \overset{?}{>} \frac{h}{4\pi}$$

$$n\sqrt{\left(\frac{1}{12}-\frac{1}{2n^2\pi^2}\right)} \overset{?}{>} \frac{1}{2\pi}$$

$$2pn\sqrt{\left(\frac{1}{12}-\frac{1}{2n^2\pi^2}\right)} \overset{?}{>} 1$$

$$\sqrt{\frac{4\pi^2n^2}{12}-\frac{4\pi^2n^2}{2n^2\pi^2}} \overset{?}{>} 1$$

$$\sqrt{\frac{1}{3}\left(\pi^2n^2-6\right)} \overset{?}{>} 1$$

$$\frac{1}{3}\left(\pi^2n^2-6\right) \overset{?}{>} 1$$

$$\left(\pi^2n^2-6\right) \overset{?}{>} 3$$

$$\pi^2n^2 \overset{?}{>} 9$$

$$\pi n \overset{?}{>} 3 \quad \text{für } n = 1 \quad \pi > 3$$
